

A NOTE ON INEXTENSIBLE FLOWS OF CURVES IN E^n

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ABSTRACT. In this paper, we investigate the general formulation for inextensible flows of curves in E^n . The necessary and sufficient conditions for inextensible curve flow are expressed as a partial differential equation involving the curvatures.

1. INTRODUCTION

It is well known that many nonlinear phenomena in physics, chemistry and biology are described by dynamics of shapes, such as curves and surfaces. The evolution of curve and surface has significant applications in computer vision and image processing. The time evolution of a curve or surface generated by its corresponding flow in -for this reason we shall also refer to curve and surface evolutions as flows throughout this article- is said to be inextensible if, in the former case, its arclength is preserved, and in the latter case, if its intrinsic curvature is preserved [8]. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of a physical applications. For example, both Chirikjian and Burdick [1] and Mochiyama et al. [10] study the shape control of hyper-redundant, or snake-like robots.

Inextensible curve and surface flows also arise in the context of many problems in computer vision [7][9] and computer animation [2], and even structural mechanics [11].

There have been a lot of studies in the literature on plane curve flows, particularly on evolving curves in the direction of their curvature vector field (referred to by various names such as “curve shortening”, flow by curvature” and “heat flow”). Particularly relevant to this paper are the methods developed by Gage and Hamilton [3] and Grayson [5] for studying the shrinking of closed plane curves to circle via heat equation.

In this paper, we develop the general formulation for inextensible flows of curves in E^n . Necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the curvatures.

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2. PRELIMINARY

To meet the requirements in the next sections, the basic elements of the theory of curves in the Euclidean n -space E^n are briefly presented in this section (A more complete elementary treatment can be found in [4][6]).

Let $\alpha : I \subset R \rightarrow E^n$ be an arbitrary curve in E^n . Recall that the curve α is said to be a unit speed curve (or parameterized by arclength functions) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of given by

$$\langle X, Y \rangle = \sum_{i=1}^n x_i y_i$$

for each $X = (x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n) \in R^n$. In particular, norm of a vector $X \in R^n$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. Let $\{V_1, V_2, \dots, V_n\}$ be the moving Frenet frame along the unit speed curve α , where $V_i (i = 1, 2, \dots, n)$ denotes the i^{th} Frenet vector field. Then Frenet formulas are given by

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-2}' \\ V_{n-1}' \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-2} \\ V_{n-1} \\ V_n \end{bmatrix}$$

where $k_i (i = 1, 2, \dots, n)$ denotes the i^{th} curvature function of the curve [4][6]. If all of the curvatures $k_i (i = 1, 2, \dots, n)$ of the curve vanish nowhere in $I \subset R$, it is called a non-degenerate curve.

3. INEXTENSIBLE FLOWS OF CURVE IN E^n

Throughout this paper, we suppose that

$$\alpha : [0, l] \times [0, w) \rightarrow E^n$$

is a one parameter family of smooth curves in E^n , where l is the arclength of the initial curve. Let u be the curve parameterization variable, $0 \leq u \leq l$. If the speed curve α is denoted by $v = \left\| \frac{d\alpha}{du} \right\|$ then the arclength of α is

$$S(u) = \int_0^u \left\| \frac{\partial \alpha}{\partial u} \right\| du = \int_0^u v du.$$

The operator $\frac{\partial}{\partial s}$ is given with respect to u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}. \quad (3.1)$$

Thus, the arclength is $ds = v du$.

Definition 3.1. Any flow of the curve can be expressed following form:

$$\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$$

where f_i denotes the i^{th} scalar speed of the curve. Let the arclength variation be

$$S(u, t) = \int_0^u v du.$$

In the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} S(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0, \quad u \in [0, l]. \quad (3.2)$$

Definition 3.2. A curve evolution $\alpha(u, t)$ and its flow $\frac{\partial \alpha}{\partial t}$ are said to be inextensible if

$$\frac{\partial}{\partial t} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0.$$

Now, we research the necessary and sufficient condition for inelastic curve flow. For this reason, we need to the following Lemma.

Lemma 3.3. Let $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$ be a smooth flow of the curve α . The flow is inextensible if and only if

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_1. \quad (3.3)$$

Proof. Since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute and $v^2 = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle$, we have

$$\begin{aligned} 2v \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle \\ &= 2 \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial t} \left(\sum_{i=1}^n f_i V_i \right) \right\rangle \\ &= 2 \left\langle v V_1, \sum_{i=1}^n \frac{\partial f_i}{\partial u} V_i + \sum_{i=1}^n f_i \frac{\partial V_i}{\partial u} \right\rangle \\ &= 2 \left\langle v V_1, \frac{\partial f_1}{\partial u} V_1 + f_1 \frac{\partial V_1}{\partial u} + \dots + \frac{\partial f_n}{\partial u} V_n + f_n \frac{\partial V_n}{\partial u} \right\rangle \\ &= 2 \left\langle v V_1, \frac{\partial f_1}{\partial u} V_1 + f_1 v k_1 V_2 + \dots + \frac{\partial f_n}{\partial u} V_n - f_n v k_{n-1} V_{n-1} \right\rangle \\ &= 2 \left(\frac{\partial f_1}{\partial u} - f_2 v k_1 \right). \end{aligned}$$

Thus, we reach

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_1.$$

□

Theorem 3.4. *Let $\{V_1, V_2, \dots, V_n\}$ be the moving Frenet frame of the curve α and $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$ be a differentiable flow of α in E^n . Then the flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = f_2 k_1. \quad (3.4)$$

Proof. Suppose that the curve flow is inextensible. From equations (3.2) and (3.3) for $u \in [0, l]$, we see that

$$\frac{\partial}{\partial t} S(u, t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\frac{\partial f_1}{\partial u} - f_2 v k_1 \right) du = 0.$$

Thus, it can be seen that

$$\frac{\partial f_1}{\partial u} - f_2 v k_1 = 0.$$

Considering the last equation and (3.1), we reach

$$\frac{\partial f_1}{\partial s} = f_2 k_1.$$

Conversely, following similar way as above, the proof is completed.

Now, we restrict ourselves to arclength parameterized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arclength s . We require the following Lemma

Lemma 3.5. *Let $\{V_1, V_2, \dots, V_n\}$ be the moving Frenet frame of the curve α . Then, the differentiations of $\{V_1, V_2, \dots, V_n\}$ with respect to t is*

$$\begin{aligned} \frac{\partial V_1}{\partial t} &= \left[\sum_{i=2}^{n-1} \left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_i \right] + \left(f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_n, \\ \frac{\partial V_j}{\partial t} &= - \left(f_{j-1} k_{j-1} + \frac{\partial f_j}{\partial s} - f_{j+1} k_j \right) V_1 + \left[\sum_{\substack{k=2 \\ k \neq i}}^n \Psi_{kj} V_k \right], \quad 1 < j < n, \\ \frac{\partial V_n}{\partial t} &= - \left(f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_1 + \left[\sum_{k=2}^{n-1} \Psi_{kn} V_k \right], \end{aligned}$$

where $\Psi_{kj} = \left\langle \frac{\partial V_j}{\partial t}, V_k \right\rangle$ and $\Psi_{kn} = \left\langle \frac{\partial V_n}{\partial t}, V_k \right\rangle$.

Proof. For $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ commute, it seen that

$$\begin{aligned}\frac{\partial V_1}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial \alpha}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial \alpha}{\partial t} \right) = \frac{\partial}{\partial s} \left(\sum_{i=1}^n f_i V_i \right) = \sum_{i=1}^n \frac{\partial f_i}{\partial s} V_i + \sum_{i=1}^n f_i \frac{\partial V_i}{\partial s} \\ &= \frac{\partial f_1}{\partial s} V_1 + f_1 \frac{\partial V_1}{\partial s} + \frac{\partial f_2}{\partial s} V_2 + f_2 \frac{\partial V_2}{\partial s} + \dots + \frac{\partial f_n}{\partial s} V_n + f_n \frac{\partial V_n}{\partial s} \\ &= \frac{\partial f_1}{\partial s} V_1 + f_1 k_1 V_2 + \frac{\partial f_2}{\partial s} V_2 + f_2 (-k_1 V_1 + k_2 V_3) + \dots + \frac{\partial f_n}{\partial s} V_n - f_n k_{n-1} V_{n-1}.\end{aligned}$$

Substituting the equation (3.4) into the last equation and using Theorem 3.4., we have

$$\frac{\partial V_1}{\partial t} = \left[\sum_{i=2}^{n-1} \left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_i \right] + \left(f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_n.$$

Now, let us differentiate the Frenet frame with respect to t for $1 < j < n$ as follows;

$$\begin{aligned}0 &= \frac{\partial}{\partial t} \langle V_1, V_j \rangle = \left\langle \frac{\partial V_1}{\partial t}, V_j \right\rangle + \left\langle V_1, \frac{\partial V_j}{\partial t} \right\rangle \\ &= \left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) + \left\langle V_1, \frac{\partial V_j}{\partial t} \right\rangle.\end{aligned}\tag{3.5}$$

From (3.5), we have obtain

$$\frac{\partial V_j}{\partial t} = - \left(f_{j-1} k_{j-1} + \frac{\partial f_j}{\partial s} - f_{j+1} k_j \right) V_1 + \left[\sum_{\substack{k=2 \\ k \neq j}}^n \Psi_{kj} V_k \right].$$

Lastly, considering $\langle V_1, V_n \rangle = 0$ and following similar way as above, we reach

$$\frac{\partial V_n}{\partial t} = - \left(f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_1 + \left[\sum_{k=2}^{n-1} \Psi_{kn} V_k \right].$$

Theorem 3.6. Suppose that the curve flow $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$ is inextensible. Then the following system of partial differential equations holds:

$$\begin{aligned}\frac{\partial k_1}{\partial t} &= f_2 k_1^2 + f_1 \frac{\partial k_1}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2 \frac{\partial f_2}{\partial s} k_2 - f_3 \frac{\partial k_2}{\partial s} - f_2 k_2^2 - f_4 k_3 k_2 \\ \frac{\partial k_{i-1}}{\partial t} &= - \frac{\partial \Psi_{(i-1)i}}{\partial s} - \Psi_{(i-2)i} k_{i-2} \\ \frac{\partial k_i}{\partial t} &= \frac{\partial \Psi_{(i-1)i}}{\partial s} - \Psi_{(i+2)i} k_{i+2} \\ \frac{\partial k_{n-1}}{\partial t} &= - \frac{\partial \Psi_{(n-1)n}}{\partial s} - \Psi_{(n-2)n} k_{n-2}.\end{aligned}$$

□

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Proof. Since $\frac{\partial}{\partial s} \frac{\partial V_1}{\partial t} = \frac{\partial}{\partial t} \frac{\partial V_1}{\partial s}$, we get

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial V_1}{\partial t} &= \frac{\partial}{\partial s} \left[\sum_{i=2}^{n-1} \left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_i + \left(f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_n \right] \\ &= \sum_{i=2}^{n-1} \left[\left(\frac{\partial f_{i-1}}{\partial s} k_{i-1} + f_{i-1} \frac{\partial k_{i-1}}{\partial s} + \frac{\partial^2 f_i}{\partial s^2} - \frac{\partial f_{i+1}}{\partial s} k_i - f_{i+1} \frac{\partial k_i}{\partial s} \right) V_i \right] \\ &\quad + \sum_{i=2}^{n-1} \left[\left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) \frac{\partial V_i}{\partial s} \right] \\ &\quad + \left(\frac{\partial f_{n-1}}{\partial s} k_{n-1} + f_{n-1} \frac{\partial k_{n-1}}{\partial s} + \frac{\partial^2 f_n}{\partial s^2} \right) V_n + \left(f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) \frac{\partial V_n}{\partial s} \end{aligned}$$

while

$$\frac{\partial}{\partial t} \frac{\partial V_1}{\partial s} = \frac{\partial}{\partial t} (k_1 V_2) = \frac{\partial k_1}{\partial t} V_2 + k_1 \frac{\partial V_2}{\partial t}.$$

Thus, from the both of above two equations, we reach

$$\frac{\partial k_1}{\partial t} = f_2 k_1^2 + f_1 \frac{\partial k_1}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2 \frac{\partial f_3}{\partial s} k_2 - f_3 \frac{\partial k_2}{\partial s} - f_2 k_2^2 - f_4 k_3 k_2.$$

For $1 < i < n$, noting that $\frac{\partial}{\partial s} \frac{\partial V_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial V_i}{\partial s}$, it is seen that

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial V_i}{\partial t} &= \frac{\partial}{\partial s} \left[- \left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_1 + \sum_{k=2}^n \Psi_{kj} V_k \right] \\ &= - \left(\frac{\partial f_{i-1}}{\partial s} k_{i-1} + f_{i-1} \frac{\partial k_{i-1}}{\partial s} + \frac{\partial^2 f_i}{\partial s^2} - \frac{\partial f_{i+1}}{\partial s} k_i - f_{i+1} \frac{\partial k_i}{\partial s} \right) V_1 \\ &\quad + \left(f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) \frac{\partial V_1}{\partial s} + \sum_{\substack{k=2 \\ k \neq i}}^n \left(\frac{\partial \Psi_{ki}}{\partial s} V_k + \Psi_{ki} \frac{\partial V_k}{\partial s} \right) \end{aligned}$$

while

$$\frac{\partial}{\partial t} \frac{\partial V_i}{\partial s} = \frac{\partial}{\partial t} (-k_{i-1} V_{i-1} + k_i V_{i+1}) = -\frac{\partial k_{i-1}}{\partial t} V_{i-1} - k_{i-1} \frac{\partial V_{i-1}}{\partial t} + \frac{\partial k_i}{\partial t} V_{i+1} + k_i \frac{\partial V_{i+1}}{\partial t}.$$

Thus, we obtain

$$\frac{\partial k_{i-1}}{\partial t} = -\frac{\partial \Psi_{(i-1)i}}{\partial s} - \Psi_{(i-2)i} k_{i-2}$$

and

$$\frac{\partial k_i}{\partial t} = \frac{\partial \Psi_{(i+1)i}}{\partial s} - \Psi_{(i+2)i} k_{i+1}.$$

Lastly, considering $\frac{\partial}{\partial s} \frac{\partial V_n}{\partial t} = \frac{\partial}{\partial t} \frac{\partial V_n}{\partial s}$ and following similar way as above, we reach

$$\frac{\partial k_{n-1}}{\partial t} = -\frac{\partial \Psi_{(n-1)n}}{\partial s} - \Psi_{(n-2)n} k_{n-2}.$$

□

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